Nonlinear Design for Inverted Pendulum using Backstepping Control Technique

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ABSTRACT
Backstepping control technique is a Lyapunov based nonlinear robust technique which is applicable to only strict feedback system. Backstepping technique has been applied on linear inverted pendulum which is under actuated system. Inverted pendulum is swung from its pendant position and stabilized at its upright position. Swing up is achieved by nonlinear backstepping controller but the linear inverted pendulum is having unstable zero dynamics which is stabilized linear backstepping controller. Linear backstepping controller stabilizes both the angle & position at upright position through integral regulated variable that transformed states of inverted pendulum into a global normal form on which backstepping design is applied and also with help of partial feedback linearization.

Keywords -Nonlinear, Linear inverted pendulum, Backstepping, Zero dynamics

1. INTRODUCTION
Inverted pendulum [1], [10] is classical example of control field in which we stabilize the pendulum at upright position from pendant position. Inverted pendulum is a highly nonlinear, under actuated and open loop unstable system. Many advanced techniques are employed such as neural, fuzzy PD, LQR, and Backstepping over conventional techniques to make system stable. In this paper, a nonlinear Backstepping [2] and Linear Backstepping are synthesized for the swing up and stabilization of inverted pendulum respectively. The backstepping is a recursive procedure in which higher order is decomposed into single order interconnected subsystem. These divided subsystems are in cascade with each other to form the overall closed loop system. The backstepping is applied on first subsystem to design a virtual control for next subsystem and design is carried out in next subsystem to design another virtual control until dynamics of subsystem contain control input explicitly.

Backstepping technique provides robustness to the system against the structured dynamics such as parameter variations and unstructured uncertainty such as un-model dynamics [6, 7]. Backstepping control technique is applicable to only strict feedback system while inverted pendulum is not strict feedback system which causes Backstepping control technique can’t be applied on all states simultaneous. If backstepping control technique is applied on linear inverted pendulum there are still some states which are not controllable these states are known as zero dynamics. If these zero dynamics are stable system then system is known as minimum phase otherwise non minimum phase system.

2. MODEL OF INVERTED PENDULUM
Linear inverted pendulum is an under actuated system which means it has more number of degree of freedom than number of actuator. That is why linear inverted pendulum is not easy to get controlled. Basically linear inverted pendulum consist of pendulum of mass ‘m’ with rod of length ‘L’ which is hinged to cart pole of mass ‘M’ mounted on rail and moves in x direction as shown in Fig. 1.

Fig. 1 Inverted Pendulum System and Free body diagram
\[
\begin{align*}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= \frac{M_0 \sin x_1 - m l x_2^2 \sin x_1 \cos x_1}{M_0 l_0 - m l \cos^2 x_1} - \frac{M_0 l_0 - m l \cos^2 x_1}{m l \cos x_1} x_3 = x_4 \\
\dot{x}_3 &= \frac{m l x_2^2 \sin x_1 \cos x_1 - x_2 \cos x_1}{M_0} \\
\dot{x}_4 &= \frac{u}{M_0} \\
\end{align*}
\]

where, states \( \{x_1, x_2, x_3, x_4\} = \{\theta, \dot{\theta}, \dot{x}, \dot{x}\} \\
M_0 = M + m l d l_0 = 1 + m l^2

### 3. CONTROLLER DESIGN

The Backstepping control design not only synthesized swing up controller but also stabilizes the pendulum at upright position however at stabilization point the zero dynamics is unstable. This two zone control strategy is achieved by linear controller and nonlinear controller based on Backstepping control technique.

#### 3.1 Synthesize of Nonlinear Swing up Controller using Backstepping:

Swing up controller is used to move pendulum from pendant position to upright position which is synthesized from system (1) by backstepping control technique in which we construct a Control Lyapunov Function (CLF) and corresponding virtual signal is determined to stabilize the subsystem until control signal comes into dynamics explicitly. There are following steps involved into controller design.

**Step 1: Stabilization of \( x_1 \) at \( x_{1d} \)**

Let new variables,

\[
\begin{align*}
e_1 &= x_1 - x_{1d} \\
\dot{e}_1 &= x_2 - \dot{x}_{1d}
\end{align*}
\]

Define CLF for \( e_1 \): Let Lyapunov function

\[
V_1 = e_1^2 / 2
\]

\[
\dot{V}_1 = e_1 \dot{e}_1 = e_1 (x_2 - \dot{x}_{1d})
\]

Let \( x_2 = x_{2d} \) be virtual control signal for which \( \dot{V}_1 \) to be negative definite

\[
x_{2d} = \dot{x}_{1d} - k_3 e_1
\]

Where \( k_3 > 0 \)

\[
- k_3 e_1^2 < 0
\]

**Step 2: Stabilization of \( x_2 \) at \( x_{2d} \)**

Let new variables,

\[
e_2 = x_2 - x_{2d}
\]

Define CLF for \( e_2 \): Let augment Lyapunov function,

\[
V_2 = e_1^2 / 2 + e_2^2 / 2
\]

\[
\dot{V}_2 = \dot{V}_1 - k_4 e_2^2 < 0
\]

\[
\dot{e}_2 = \dot{x}_2 - \dot{x}_{2d} = -k_4 e_2
\]

So from (1), and (7), Nonlinear Feedback control law for swing up,

\[
u = (M_0 l_0 - (m l \cos x_1)^2) / (m l \cos x_1) \{k_4 (x_2 - x_{2d})
\]

\[
- \dot{x}_{2d} = \frac{m g l M_0 \sin x_1 + (m l x_2^2 \sin x_1 \cos x_1)}{(M_0 l_0 - (m l \cos x_1)^2)}
\]

So derivative of Lyapunov function for system (1), given by (5), is made negative definite by control law and virtual control law given by (3), and (8), which results into \( e_1 = 0 \) then \( x_1 = x_{1d} \) and for swing up \( x_{1d} = \pi \).

### A. Linear Controller:

At stabilization point \( \theta = \pi \), put \( \dot{\theta} = 0 \) and \( u=0 \) in (1), to determine stability of zero dynamics,

\[
\dot{x}_4 = \frac{-m g l \sin x_1 - m l \cos x_1 \dot{x}_2}{m l \cos x_1} x_2
\]

\[
x_4 = g \tan x_1
\]

As we know, \( \sin x_1 \cong -x_1 \cos x_1 = -1 \) and hence,

\[
x_4 = g x_1
\]

which is unstable zero dynamics. This zero dynamics is stabilized by linear backstepping controller. Linear backstepping controller has been synthesized with help of regulated variable by using linear model of inverted pendulum to stabilize the angle as well as position at stabilizing point. The linear model of inverted pendulum is represented around equilibrium point by equation,

\[
\begin{align*}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= - \frac{M_0 l_0}{m l l_0} \dot{x}_4 + \frac{m l u}{m l l_0} \\
\dot{x}_3 &= x_4 \\
\dot{x}_4 &= \frac{m l \beta x_4}{m l l_0} - \frac{l_0 u}{m l l_0}
\end{align*}
\]

where \( M_0 = M + m \) and \( l_0 = l + m l^2 \)

In addition, partial feedback linearization technique \[8\] is applied to reduce the complexity. Let the control law,

\[
u = \{M_0 - m l l_0 \} \dot{u} + m l \beta x_4 / l_0 \]

where \( u \) is the control input which convert equation into following equation,

\[
\begin{align*}
\dot{x}_1 &= x_2 \dot{x}_2 = \dot{u} \\
\dot{x}_3 &= x_4 \dot{x}_4 = \beta x_4 / l_0 - m l \dot{u} / l_0
\end{align*}
\]
Now backstepping control technique is applied on (10), and (12), to synthesize controller with help of regulated variable in following three steps so as to stabilize the zero dynamics.

Step 1: Introduce regulated variable $q_1$ and $q_2$
Let variable $q_1$ and $q_2$,

\[ q_1 = x_1 + k_1x_3 + k_2[mlx_1 + l_0x_3] \]

\[ q_2 = \int q_1 dt \]

\[ \dot{q}_1 = x_2 + k_1x_4 + k_2\beta x_3 \]
Define CL for $q_1$: Let Lyapunov function,

\[ V_3 = \frac{1}{2} q_1^2 \text{ then } \dot{V}_3 = q_1 \dot{q}_1 \]

(13)

$q_2d$ is the virtual control for which $\dot{V}_3$ will become negative definite then,

\[ q_{1d} = -k_1x_4 + k_2\beta x_1 - c_1q_1 - c_2q_2 \]
(14)

Where $c_1$ and $c_3 > 0$.

\[ \dot{q}_1 = x_2 - q_{1d} \]
\[ \dot{q}_1 = q_3 - c_1q_1 - c_2q_2 \]
(15)

Step 2: Stabilize regulated variable $q_2$
Define CL for $q_2$: Let augment Lyapunov function,

\[ V_4 = q_1^2/2 + c_2q_2^2/2 \]
\[ \dot{V}_4 = q_1\dot{q}_1 + c_2q_2\dot{q}_2 \]
(16)

From (15), and (16),

\[ \dot{V}_4 = -c_1q_1^2 + q_1q_3 \]

Step: 3 Stabilize regulated variable $q_3$,

\[ q_3 = x_2 - q_{1d} \]
\[ \dot{q}_3 = u_v + k_1[\beta x_1/l_0 - mlv/l_0] + k_2\beta x_2 \]

\[ + c_1[q_3 - c_1q_1 - c_2q_2] - c_2q_1 \]
Define CL for $q_3$: Let augment Lyapunov function,

\[ V_5 = V_4 + \frac{1}{2} q_3^2 \]
(17)

Now, for $\dot{V}_5$ to be negative definite,

\[ u_v = d_3d_1q_1 + d_2q_3 + k_1\beta x_1/l_0 \]
\[ + k_2\beta x_2 - c_1c_2q_2 \]
(18)

where,

\[ d_1 = (1 + c_2 - c_1^2) \]
\[ d_3 = l_0/l_0 - k_1(ml)^2 \]

\[ \dot{V}_5 = -c_1q_1^2 - c_3q_3^2 \]
Where $c_1$, $c_2$, $c_3 > 0$
So linear controller for stabilization from (11),

\[ u = d_3d_4[d_1q_1 + d_2q_3 + k_1\beta x_1/l_0 \]
\[ + k_2\beta x_2 - c_1c_2q_2] \]
(19)

4. SIMULATION
Simulation results of real time implementation are shown in figure with initial condition $x_1(0) = 0$ and $x_3(0) = 0$. System parameter and designed parameters are as shown in Table 1 and Table 2 as shown below.

![Fig 2 Swing up of inverted pendulum](image)

![Fig 3 Stabilization of pendulum through integral regulated variable](image)

![Fig 4 Stabilization of inverted pendulum using regulated Variable](image)

![Fig 5 Unstable zero dynamics](image)

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The inverted pendulum is swung up by nonlinear backstepping controller as shown in Fig. 2 within 8sec without considering friction which causes it more time to swing up. The stabilization of inverted pendulum at upright position is controlled by linear backstepping control using regulated variable and integral regulated variable as shown in Fig. 3 and Fig. 4 which shows that integral regulated variable minimizes the steady. At stabilization, zero dynamics are represented by,

\[
\begin{bmatrix}
\dot{x}_3 \\
\dot{x}_4
\end{bmatrix} = 
\begin{bmatrix}
0 & 1 \\
-p1 & -p2
\end{bmatrix}
\begin{bmatrix}
x_3 \\
x_4
\end{bmatrix}
\]

where,

\[ p1 = d_3(d_1 + d_2c_1)k_1 \]

and

\[ p2 = d_3[(d_1k_2 + d_2k_1 + d_2c_1k_2ml)] \]

So for zero dynamics to be stable,

\[ d_3 > 0, \quad 0 < k_1 < \frac{b}{(ml)^2} = 2.6 \]

If \( k_1 = 2.8 \) the system becomes unstable which is shown in Fig. 5. Controller able to accommodate the disturbance and stabilize the pendulum at equilibrium point which is shown in Fig. 6 when disturbance is introduced in system at \( t=12.5 \) sec which shows robustness of controller for bounded disturbance.

### 5. CONCLUSION

Inverted pendulum is controlled by two zone control strategy known as Swing up controller and stabilization controller. Inverted Pendulum which have unstable zero dynamics at upright has been stabilized by linear controller using integral regulated variables while swing up controller is synthesized by nonlinear backstepping control technique. From simulation results it is evident that Backstepping control technique is very robust irrespective of parametric or un-modeled dynamics. Analytical complexity is increased as the order of system is increased and also saturation problem is occurred due to derivative of virtual control.

### REFERENCES


