Robust Pole Placement using Linear Quadratic Regulator 
Weight Selection Algorithm

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ABSTRACT
The main advantage of pole placement technique is that it places all the poles at desired location using state feedback gain matrix. Using feedback, the poles of the system can be shifted so we can shape the closed loop characteristics of system to meet the design requirement. Even though pole placement method can give the desired characteristic but it does not guarantee a robust system. So controller design using pole placement may not be insensitive to system parameter variations and external disturbance. The linear quadratic regulator (LQR) is an optimal design technique that guarantees a robust system. The difficulty lies in choosing a weighting matrix for the LQR cost function that gives the desired poles. In this paper, an algorithm is developed that finds the location of poles which satisfy the desired goal and give a robust system also. The LQR method guarantees robustness, but not allows pole placement in a specific regions, and the pole placement gives the desired performance but not guarantees robustness. Therefore it may not be possible to achieve the desired performance with robustness.

This paper present a technique, when the desired poles are outside the allowable LQR region, a gradient search method is used to find the achievable pole inside the region that is closest to the desire poles. This technique is accurate for selecting robust poles at or close to the desired pole locations. To achieve this system should be completely controllable. This paper is only concerned with closed loop system and is targeted to find the amount of internal feedback that will bring the actual performance closer to the intended performance of a system along with robustness.

II. BACKGROUND OF POLE PLACEMENT AND LQR
In pole placement technique instead of using controllers with fixed configuration in the forward path or feedback path, control is achieved by feeding back the state variables through real constant under the restriction that system is completely controllable [5].

Consider a linear dynamic system in the state space form

\[ \dot{x} = Ax + Bu \]  
\[ y = Cx \]

If the system is completely controllable we can use pole placement for stabilizing the system or improving its transient response. Here we represent control signal \( u \) as a linear combination of the state variables, that is

\[ u = -Kx \]

where \( K = [K_1 \ K_2 \ K_3 \ K_4] \) state feedback gain matrix. Now closed-loop system, given by

\[ \dot{x} = (A - BK)x \]  
\[ y = Cx \]
A problem arises when a control is required to drive the plant output from a nonzero state to zero state for a plant subjected to unwanted disturbances. The problem is to drive the system back to the zero state as fast as possible while trying to use the smallest control inputs. Anderson and Moore define this as a regulator problem. The optimal solution is to find a control input $u$ to minimize the quadratic cost function

$$J = \int_0^\infty (x^T Q x + u^T R u) dt$$

(6)

where $Q$ and $R$ are weighting matrices chosen by the designer. This is the classical linear quadratic regulator design problem. An advantage of using LQR design is that the system will always be stable and robust. The optimal solution is given by

$$K = R^{-1} B^T P$$

(7)

where matrix $P$ is the positive-definite solution (which is unique) of the algebraic Riccati equation

$$PA + A^T P + Q - PBR^{-1} B^T P = 0$$

(8)

Where $Q$ and $R$ are positive-definite symmetric real matrix. LQR method guarantees robustness, but only allows pole placement in a specific region, and the poles that give the desired performance may or may not be in these regions. Therefore it may not be possible to use this method to achieve both robustness and exact pole placement.

A system is completely controllable if any initial state $x_0$ can be shifted to any final state $x_f$ in a finite time. In other words, a feedback matrix exists that can shift a closed loop pole to any desired location. Controllability for the linear system given by

$$M = \begin{bmatrix} B & AB & A^2 B & \ldots & A^{n-1} B \end{bmatrix}$$

(8)

where $n$ is the number of states. The system is controllable if the rank of $M$ is $n$.

A system is detectable if all the unstable (right half plane) poles are observable, and a system is stabilizable if all the unstable (right half plane) poles are controllable.

### III. ROBUST POLE PLACEMENT ALGORITHM

It may not be possible achieve both robustness and exact pole placement using pole placement technique. In that case a choice would have to be made. This paper takes the robustness has higher priority over pole placement and hence uses the LQR technique to choose the poles. When the desired pole is outside the allowable LQR region, a gradient search is used to find the achievable pole inside the region that is the closest to the desired pole. This is done by minimizing the cost function

$$J^* = \sum_{i=1}^n [V_i(\lambda_{des i} - \lambda_{ach i})^2]$$

(9)

where

- $V_i$ is a positive definite weighting parameter for the poles,
- $\lambda_{des i}$ is the $i^{th}$ desired pole, and
- $\lambda_{ach i}$ is the $i^{th}$ achievable pole from LQR

Thus, when $J$ is made small, the system comes close to satisfying the desired pole range and is simultaneously robust. Pole weighting is included in the cost function to give priority to poles that need to be a specific value. Some poles may have limitations that prevent deviations from the desired locations. For example, an actuator may have characteristics that determine the location of the pole, and if a designer were to move the pole he would violate the physical model.

*Starting the second time through, check to see if the magnitude of $\frac{\partial J}{\partial H}$, $\frac{\partial J}{\partial M}$ and $|J^*_K - J^*_K-1|$ are less than a prescribed tolerance (where $K$ is the iteration number). To circumvent this, a high weighting is placed on that particular Eigen value difference in the cost function. Although the algorithm may not give the exact overall desired pole spectrum, it gives the designer the ability to enforce a required pole placement. The algorithm also delivers a system that is robust, stable and close to the required pole specifications. The goal of the algorithm is to minimize the cost function given in equation 9. Where the pole weights $V_i$ are chosen by the designer. In the LQR problem, the state weighting matrix is required to be positive semi definite ($Q \geq 0$) and the control weighting matrix positive definite ($R > 0$). $Q$ can be made positive semi definite by defining $Q = H^T H$. Likewise, $R$ can be made positive definite by defining $R=M^T M$ and requiring $M$ to be square or tall (i.e. more rows) and have full column rank.

Fig. 1: Block Diagram of pole placement
Fig 2: Flow chart of algorithm to find robust pole locations

IV. SIMULATION AND RESULTS

The equation of motion for the system shown in figure 3 is given by

\[ u = m \ddot{\alpha} \]  

\[ \dot{\alpha} \]

\[ \alpha \]

\[ m \]

\[ u \]

\[ \ddot{\alpha} \]

\[ (10) \]

Fig 3: Second Order SISO system

So the state space representation is given as

\[ \begin{bmatrix} \dot{\alpha} \\ \ddot{\alpha} \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \alpha \\ \ddot{\alpha} \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{m} \end{bmatrix} u \]  

\[ (11) \]

Taking \( m = 1 \), the system is given as

\[ \begin{bmatrix} \dot{\alpha} \\ \ddot{\alpha} \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \alpha \\ \ddot{\alpha} \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u \]  

\[ (12) \]

So the characteristic equation is given as

\[ 1 + G(s)H(s) = 0 \]  

\[ (13) \]

Characteristic equation of full state feedback system is given as

\[ 1 + K[sI - A]^{-1}B = 0 \]  

\[ (14) \]

where \( K \) is given as

\[ K = [K_1 \ K_2] \]  

\[ (15) \]

Now the characteristic equation of system is

\[ 1 + \frac{K_1}{s^2} + \frac{K_2}{s} = 0 \]  

\[ (16) \]

or

\[ s^2 + K_2 s + K_1 = 0 \]  

\[ (17) \]

A typical second order system has the form

\[ s^2 + 2\zeta \omega_n s + \omega_n^2 = 0 \]  

\[ (18) \]

Phase Margin (PM) of a typical second order system is given as

\[ PM = \tan^{-1} \frac{2\zeta [2\zeta^2 + (4\zeta^2 + 1)^{\frac{1}{2}}]^{\frac{1}{2}}}{1} \]  

\[ (19) \]

The mathematical constraint on a Nyquist plot from LQR design is
\[ |1 + GH| \geq 1 \]  

(20)

Applying this constraint for above system we find that the damping ratio must meet

\[ \zeta \geq 0.7071 \]  

(21)

This gives the following system characteristics

<table>
<thead>
<tr>
<th>Peak Overshoot (%)</th>
<th>Rise Time (sec)</th>
<th>Settling Time (sec)</th>
<th>( \zeta )</th>
<th>( \omega_n )</th>
<th>PM</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.04 %</td>
<td>0.51</td>
<td>1.87</td>
<td>0.707</td>
<td>3.54</td>
<td>65.5</td>
</tr>
</tbody>
</table>

The above PM can be verified using Bode plot. PM is greater than 60° which is verified using LQR that minimum PM required is 60°.

The same result is verified using the algorithm in MATLAB which shows that algorithm is able to accurately find \( \lambda_{ach} \).

<table>
<thead>
<tr>
<th>( \lambda_{ach} )</th>
<th>K</th>
<th>Q</th>
</tr>
</thead>
<tbody>
<tr>
<td>Algorithm</td>
<td>-2.5+2.5j, -2.5-2.5j</td>
<td>[12.5 5]</td>
</tr>
<tr>
<td>Calculated</td>
<td>-2.5+2.5j, -2.5-2.5j</td>
<td>[12.5 5]</td>
</tr>
</tbody>
</table>

VI. CONCLUSION

This paper presents a technique to find the robust pole location for pole placement technique to calculate the feedback gain matrix. Even though the pole placement method can give the desired performance characteristic, it does not guarantee a robust system. That is, one that is insensitive to system parameter variations and external disturbances. The LQR method gives the optimal solution considering the control signal \( u \) by minimizing the quadratic cost function. An advantage of using LQR design technique is that system will always be stable and robust. The LQR method guarantees robustness, but only allows pole placement in a specific region.

Using the technique presented in this paper we can find the robust pole location which gives a PM of minimum 60°. This result is verified using the given algorithm in the paper.

VI. REFERENCES

Books:


Journal Papers:


