A Common Coupled Fixed Point Theorem in Complex Valued Metric Space for Two Mappings, Satisfying A Rational Inequality

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ABSTRACT
In 2006 Bhaskar and Laxmi Kantam introduced the concept of coupled fixed point for a partially ordered set. In this paper a common coupled fixed point result for two mappings are presented.

Key Words: Complex valued metric space, coupled fixed point.

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I. INTRODUCTION

One of the main pillar in the study of fixed point theory is Banach Contraction principle which was done by Banach in 1922. Fixed point theory is very useful in various branches of mathematics and science.

In 2011 Akbar Azam et. al [1] introduced the concept of complex valued metric space. The concept of coupled fixed point was first introduced by Bhaskar and Laxikantham [2] in 2006. Recently some researchers prove some coupled fixed point theorems in complex valued metric space in [4], [5]. The main purpose of this paper is to obtain a common coupled fixed point result in complex valued metric space.

Let \( \mathbb{C} \) be the set of all complex numbers

and \( z_1, z_2 \in \mathbb{C} \). Define a partial order relation \( \preceq \) on \( \mathbb{C} \) as follows:

\[ z_1 \preceq z_2 \text{ if and only if } \text{Re}(z_1) \leq \text{Re}(z_2) \text{ and } \text{Im}(z_1) \leq \text{Im}(z_2). \]

Thus \( z_1 \preceq z_2 \) if one of the followings holds:

1. \( \text{Re}(z_1) = \text{Re}(z_2) \) and \( \text{Im}(z_1) = \text{Im}(z_2) \),
2. \( \text{Re}(z_1) < \text{Re}(z_2) \) and \( \text{Im}(z_1) = \text{Im}(z_2) \),
3. \( \text{Re}(z_1) = \text{Re}(z_2) \) and \( \text{Im}(z_1) < \text{Im}(z_2) \),
4. \( \text{Re}(z_1) < \text{Re}(z_2) \) and \( \text{Im}(z_1) < \text{Im}(z_2) \).

We write \( z_1 \preceq z_2 \) if \( z_1 \preceq z_2 \) and \( z_1 \neq z_2 \) i.e., one of (2), (3) and (4) is satisfied and we will write \( z_1 \prec z_2 \) if only (4) is satisfied.

Remark 1: We can easily check the followings:

(i) \( a, b \in \mathbb{R}, a \leq b \Rightarrow az \preceq bz \ \forall z \in \mathbb{C} \).
(ii) \( 0 \preceq z_1 \preceq z_2 \Rightarrow |z_1| < |z_2| \).
(iii) \( z_1 \preceq z_2 \) and \( z_2 \prec z_3 \Rightarrow z_1 \prec z_3 \).

Azam et al. [1] defined the complex valued metric space in the following way:

Definition 1 ([1]): Let \( X \) be a nonempty set. Suppose that the mapping \( d : X \times X \to \mathbb{C} \) satisfies the following conditions:

(C1) \( 0 \preceq d(x, y), \) for all \( x, y \in X \) and \( d(x, y) = 0 \) if and only if \( x = y \);

(C2) \( d(x, y) = d(y, x) \) for all \( x, y \in X \);

(C3) \( d(x, y) \preceq d(x, z) + d(z, y) \) for all \( x, y, z \in X \).

Then \( d \) is called a complex valued metric on \( X \) and \( (X, d) \) is called a complex valued metric space.
Example 1.1: Let $X = \mathbb{R}$. Define the mapping $d: X \times X \to \mathbb{C}$ by
\[ d(x, y) = \log |x - y|, \forall \ x, y \in \mathbb{R}, \]
where $\mathbb{C}$ is a fixed complex number, such that $0 < \arg(z) < \frac{\pi}{2}$ and $|z| > 1$ [Here logarithm takes only the principle value].

Then clearly we can show that $(X, d)$ is a complex valued metric space.

Definition 2 ([1]): Let $(X, d)$ be a complex valued metric space. Then

(i) A point $x \in X$ is called an interior point of a set $A \subseteq X$ if there exists $0 < r \in \mathbb{C}$ such that $B(x, r) = \{y \in X : d(x, y) < r\} \subseteq A$.

A subset $A \subseteq X$ is called open if each element of $A$ is an interior point of $A$.

(ii) A point $x \in X$ is called a limit point of $A \subseteq X$ if for every $0 < r \in \mathbb{C}$,
\[ B(x, r) \cap (A - \{x\}) \neq \emptyset. \]

A subset $A \subseteq X$ is called closed if each element of $X - A$ is not a limit point of $A$.

(iii) The family $F = \{B(x, r) : x \in X, 0 < r\}$ is a sub-basis for a Hausdorff topology $\tau$ on $X$.

Definition 3 ([1]): Let $(X, d)$ be a complex valued metric space. Then

(i) A sequence $\{x_n\}$ in $X$ is said to converge to $x \in X$ if for every $0 < r \in \mathbb{C}$ there exists $N \in \mathbb{N}$ such that $d(x_n, x) < r, \forall \ n > N$.

We denote this by $\lim_{n \to \infty} x_n = x$ or $x_n \to x$ as $n \to \infty$.

(ii) If for every $0 < r \in \mathbb{C}$ there exists $N \in \mathbb{N}$ such that $d(x_n, x_{n+m}) < r$ for all $n > N, m \in \mathbb{N}$, then $\{x_n\}$ is called a Cauchy sequence in $(X, d)$.

(iii) If every Cauchy sequence in $X$ is convergent in $X$ then $(X, d)$ is called a complete complex valued metric space.

Definition 4[2]: An element $(x, y) \in X \times X$ is called a coupled fixed point of the mapping $T: X \times X \to X$ if $x = T(x, y)$ and $y = T(y, x)$.

Example 4.1: Let $X = \mathbb{R}$ and $T: X \times X \to X$ defined as $T(x, y) = x^2y^3$. Then $(0, 0)$ and $(1, 1)$ are two coupled fixed point of $T$.

Lemma 1 ([1]): Let $(X, d)$ be a complex valued metric space and $\{x_n\}$ be a sequence in $X$. Then $\{x_n\}$ converges to $x \in X$ if and only if $|d(x_n, x)| \to 0$ as $n \to \infty$.

Lemma 2 ([1]): Let $(X, d)$ be a complex valued metric space and $\{x_n\}$ be a sequence in $X$. Then $\{x_n\}$
is a Cauchy sequence if and only if 
\[ |d(x_n, x_{n+m})| \to 0 \text{ as } n \to \infty \text{ where } m \in \mathbb{N}. \]

**Lemma 3 ([3])**: Let \((X, d)\) be a complex valued metric space and \(\{x_n\}\) be a sequence in \(X\) such that 
\[ \lim_{n \to \infty} x_n = x. \] Then for any \(a \in X\), 
\[ \lim_{n \to \infty} d(x_n, a) = d(x, a). \]

**II. MAIN RESULT.**

In this section we present the main result.

**Theorem**: Let \((X, d)\) be a complete complex valued metric space. Let \(S, T : X \times X \to X\), such that
\[ d(S(x, y), T(u, v)) \leq a \frac{d(x, u) + d(y, v)}{2} + b \frac{d(x, y) + d(u, v)}{2} \]
\[ , \forall x, y, u, v \in X , \]
where \(a\) and \(b\) are non-negative integers with 
\[ a + b < 1. \]

Then \(S, T\) have a unique common coupled fixed point in \(X \times X\).

**Proof**: Let \(x_0, y_0 \in X\) be arbitrary.

We define two sequences \(\{x_n\}, \{y_n\}\) as
\[ x_{2k+1} = S(x_{2k}, y_{2k}), x_{2k+2} = T(x_{2k+1}, y_{2k+1}) \]
\[ y_{2k+1} = S(y_{2k}, x_{2k}), y_{2k+2} = T(y_{2k+1}, x_{2k+1}) \]

Now,
\[ d(x_{2k+1}, x_{2k+2}) = d(S(x_{2k}, y_{2k}), T(x_{2k+1}, y_{2k+1})) \]
\[ \leq a \frac{d(x_{2k}, x_{2k+1}) + d(y_{2k}, y_{2k+1})}{2} + b \frac{d(x_{2k}, y_{2k}) + d(y_{2k}, x_{2k+1})}{2} \]

Thus
\[ (2a-b)d(x_{2k+1}, x_{2k+2}) \leq a[d(x_{2k}, x_{2k+1}) + d(y_{2k}, y_{2k+1})] + bd(x_{2k}, x_{2k+1}) \]

Therefore
\[ d(x_{2k+1}, x_{2k+2}) \leq \frac{a+b}{2a-b} d(x_{2k}, x_{2k+1}) + \frac{a}{2a-b} d(y_{2k}, y_{2k+1}) \]

Similarly we can show that,
\[ d(y_{2k+1}, y_{2k+2}) \leq \frac{a+b}{2a-b} d(y_{2k}, y_{2k+1}) + \frac{a}{2a-b} d(x_{2k}, x_{2k+1}) \]

Adding (1) and (2) we get,
\[ d(x_{2k+1}, x_{2k+2}) + d(y_{2k+1}, y_{2k+2}) \leq \frac{a+b}{2a-b} [d(x_{2k}, x_{2k+1}) + d(y_{2k}, y_{2k+1})] \]

Now, we take \(h = \frac{2a+b}{2a-b}\). Then \(0 \leq h < 1\), since 
\[ 0 \leq a + b < 1. \]

Thus,
\[ d(x_{2k+1}, x_{2k+2}) + d(y_{2k+1}, y_{2k+2}) \leq h[d(x_{2k}, x_{2k+1}) + d(y_{2k}, y_{2k+1})] \]

Similarly, we can also show that,
\[ d(x_{2k+2}, x_{2k+3}) + d(y_{2k+2}, y_{2k+3}) \leq h[d(x_{2k+1}, x_{2k+2}) + d(y_{2k+1}, y_{2k+2})] \]

Now for \(n \in \mathbb{N}\), we have,
\[ d(x_{n+2}, x_{n+1}) + d(y_{n+2}, y_{n+1}) \leq h[d(x_{n+1}, x_{n}) + d(y_{n+1}, y_{n})] \]
\[ \leq h^2[d(x_{n}, x_{n-1}) + d(y_{n}, y_{n-1})] \]
\[ \leq \ldots \ldots \ldots \ldots \ldots \ldots \]
Now, for \( m > n \),
\[
\begin{align*}
d(x_m, x_n) + d(y_m, y_n) & \leq [d(x_n, x_{n+1}) + d(y_n, y_{n+1})] + [d(x_{n+1}, x_{n+2}) + d(y_{n+1}, y_{n+2})] + \ldots + [d(x_m, x_{m+1}) + d(y_m, y_{m+1})] \\
& \leq [d(x_n, x_{n+1}) + d(y_n, y_{n+1})] + [d(x_{n+1}, x_{n+2}) + d(y_{n+1}, y_{n+2})] + \ldots + [d(x_{m-1}, x_m) + d(y_{m-1}, y_m)].
\end{align*}
\]
\[
\begin{align*}
& \leq h^n + h^{n+1} + h^{n+2} + \ldots + h^{m-1} [d(x_1, x_0) + d(y_1, y_0)] \\
& \leq \frac{h^n}{1-h} [d(x_1, x_0) + d(y_1, y_0)] \\
& \rightarrow 0 \text{ as } n \rightarrow \infty.
\end{align*}
\]
Thus \( d(x_m, x_n) \rightarrow 0 \) and \( d(y_m, y_n) \rightarrow 0 \) as \( m, n \rightarrow \infty \).

Thus \( \{x_n\} \) and \( \{y_n\} \) are Cauchy sequences.

Since \( X \) is complete, there exists \( x, y \in X \), such that, \( x_n \rightarrow x \) and \( y_n \rightarrow y \), as \( n \rightarrow \infty \).

Now,
\[
\begin{align*}
d(S(x, y), x) & \leq d(S(x, y), x_{2k+1}) + d(x_{2k+2}, x) \\
& = d(S(x, y), T(x_{2k+1}, y_{2k+1})) + d(x_{2k+2}, x) \\
& \leq \frac{a}{b} \left[ d(x, x_{2k+1}) + d(y, y_{2k+1}) \right] + \frac{b}{d(x, x_{2k+1}) + d(y, y_{2k+1})} \left[ d(x_{2k+2}, x) \right]
\end{align*}
\]

Letting \( k \rightarrow \infty \) and using Lemma - 3, we get,
\[
d(S(x, y), x) \leq \frac{d(S(x, y), x)}{2}
\]
Thus \( \left( 1 - \frac{b}{2} \right) |d(S(x, y), x)| \leq 0 \).

Since \( 0 \leq a + b < 1 \), \( d(S(x, y), x) = 0 \) and hence \( S(x, y) = x \).

Similarly, we can show that, \( S(y, x) = y \).

Again,
\[
d(x, T(x, y)) = d(S(x, y), T(x, y))
\]
\[
\begin{align*}
& \leq \frac{a}{b} \left[ d(x, y) + d(x, x_{2k+1}) + d(y, y_{2k+1}) \right] + \frac{b}{d(x, x_{2k+1}) + d(y, y_{2k+1})} \left[ d(x_{2k+2}, x) \right]
\end{align*}
\]

Thus \( \left( 1 - \frac{b}{2} \right) |d(x, T(x, y))| \leq 0 \).

Therefore \( d(x, T(x, y)) = 0 \) and hence \( T(x, y) = x \).

Similarly, we can show that, \( T(y, x) = y \).

Thus,
\[
S(x, y) = T(x, y) = x \quad \text{and} \quad S(y, x) = T(y, x) = y.
\]

Therefore \( (x, y) \) is a common coupled fixed point of \( S \) and \( T \).

For uniqueness, let \( (x^*, y^*) \in X \times X \) such that,
\[
S(x^*, y^*) = T(x^*, y^*) = x^* \quad \text{and} \quad S(y^*, x^*) = T(y^*, x^*) = y^*.
\]
Now, \( d(x, x^*) = d(S(x, y), T(x^*, y^*)) \)
\[
\leq a \frac{d(x, x^*) + d(y, y^*)}{2} + b \frac{d(S(x, y) + d(x^*, y^*))}{2}
\]
\[
= a \frac{d(x, x^*) + d(y, y^*)}{2}
\] ............................... (3)

Similarly, we can show that,
\[
d(y, y^*) \leq a \frac{d(x, x^*) + d(y, y^*)}{2}
\] ............................... (4)

Adding (3) and (4), we get,
\[
d(x, x^*) + d(y, y^*) \leq a[d(x, x^*) + d(y, y^*)]
\]
Thus, \( (1 - a)[d(x, x^*) + d(y, y^*)] \leq 0. \)
Therefore, \( d(x, x^*) + d(y, y^*) = 0 \)
Thus \( d(x, x^*) = 0 \) and \( d(y, y^*) = 0 \) and hence \( x = x^* \) and \( y = y^* \).

i.e. \( (x, y) = (x^*, y^*) \).

Thus \( (x, y) \) is the unique common fixed point of \( S \) and \( T \).

**Example:** Let \( X = [0,1] \) with complex valued metric \( d: X \times X \to X \), such that
\[
d(x, y) = |x - y|, \forall x, y \in X.
\]
We define two mappings \( S, T: X \times X \to X \), as
\[
S(x, y) = \frac{xy}{4} \text{ and } T(x, y) = \frac{xy}{3}.
\]
Then
\[
d(S(x, y), T(u, v)) \leq a \frac{d(x, x^*) + d(y, y^*)}{2} + b \frac{d(S(x, y)) + d(T(u, v))}{2}, \forall x, y, u, v \in X,
\]
for \( a = \frac{9}{10} < 1 \) and \( 0 \leq b < \frac{1}{10} \). Thus here
\[
a + b < 1.
\]
Thus the mappings \( S \) and \( T \) satisfy the required condition for the theorem.
Here we see that \( (0,0) \) is the unique common coupled fixed point of \( S \) and \( T \).

**REFERENCES**


