\( \alpha \) - lacunary \( \Delta \) - statistically convergent in fuzzy soft real numbers

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ABSTRACT

In this paper we introduced the space of \( \Delta \) - bounded and \( \Delta \) - convergent difference sequence of fuzzy soft real numbers and the space of \( \alpha \) - lacunary \( \Delta \) - statistically convergent different sequence of fuzzy soft real numbers relating these concepts some theorem are derive.

Keyword - Fuzzy soft real number, \( \Delta \) - bounded, \( \Delta \) - convergent and \( \alpha \) - lacunary \( \Delta \) - statistically convergent.

I. INTRODUCTION

The concept of fuzzy sets was first introduced by Zadeh[1]. Bounded and convergent sequence of fuzzy numbers were introduced by Matloka[11]. Matloka show that every convergent sequence of fuzzy numbers is bounded. Later on sequence of fuzzy numbers have been discussed by Nanda[13], Nuray and Savaş[15], Nuray [16], Kwon[8], Savaş [17], Bilgin [2] Basarır and Mursaleen [1,12], Fang and Huang [4] and many others. The concept of statistical convergence was introduced Fast [5]. Schoenberg [18] studied statistical convergence as a summability method and listed some of elementary properties of statistical convergence.

As an extension we have defined new concepts \( \Delta \) - convergent, \( \Delta \) - bounded and \( \alpha \) - lacunary \( \Delta \) - statistically convergent sequence in the space of fuzzy soft real numbers.

II. PRELIMINARIES

In this section we present some basic definitions of fuzzy soft set. Throughout our discussion, \( U \) refers to an initial universe, \( E \) the set of all parameters for \( U \) and \( P(\tilde{U}) \) the set of all fuzzy sets of \( U \). \( (U, E) \) means the universal set \( U \) and the parameter set \( E \).

Definition 2.1 [6]
A pair \((F, E)\) is called a soft set (over \( U \)) if and only if \( F \) is a mapping of \( E \) into the set of all subsets of the set \( U \).

In otherwords, the soft set is a parameterized family of subsets of the set \( U \). Every set \( F(\varepsilon), \varepsilon \in E \), from this family may be considered as the set of \( \varepsilon \) elements of the soft set \((F, E)\), or as the set of \( \varepsilon \) - approximate elements of the soft set.

Definition 2.2 [8]
A pair \((F, A)\) is called a fuzzy soft set over \( U \) where \( F : A \to P(\tilde{U}) \) is a mapping from \( A \) into \( P(\tilde{U}) \).

Definition 2.3 [8]
For two fuzzy soft sets \((F, A)\) and \((G, B)\) in a fuzzy soft class \((U, E)\), we say that \((F, A)\) is a fuzzy soft subset of \((G, B)\), if

(i) \( A \subseteq B \)
(ii) For all \( \varepsilon \in A, F(\varepsilon) \subseteq G(\varepsilon) \) and is written as \((F, A) \preceq (G, B)\).

Definition 2.4 [8]
Union of two fuzzy soft sets \((F, A)\) and \((G, B)\) in a soft class \((U, E)\) is a fuzzy soft set \((H, C)\) where \( C = A \cup B \) and \( \forall \varepsilon \in C, H(\varepsilon) = \begin{cases} F(\varepsilon), & \text{if } \varepsilon \in A - B \\ G(\varepsilon), & \text{if } \varepsilon \in B - A \end{cases} \) and is written as \((F, A) \cup (G, B) = (H, C)\).

Definition 2.5 [8]
Intersection of two fuzzy soft sets \((F, A)\) and \((G, B)\) in a soft class \((U, E)\) is a fuzzy soft set \((H, C)\) where \( C = A \cap B \) and \( \forall \varepsilon \in C, H(\varepsilon) = F(\varepsilon) \) or \( G(\varepsilon) \) (as both are same fuzzy set) and is written as \((F, A) \cap (G, B) = (H, C)\).

Definition 2.6 [13]
Let \( A \subseteq E \) then the mapping \( F_A : E \to \tilde{P}(U) \), defined by \( F_A(\varepsilon) = \mu^\varepsilon F_A \) (a fuzzy subset of \( U \)), is called soft set over \((U, E)\), where \( \mu^\varepsilon F_A = \tilde{0} \) if \( e \in E - A \) and \( \mu^\varepsilon F_A \neq \tilde{0} \) if \( e \in A \). The set of all fuzzy soft set over \((U, E)\) is denoted by \( FS(U, E) \).

Definition 2.7 [13]
The fuzzy soft set \( F_\emptyset \in FS(U, E) \) is called null fuzzy soft set and it is denoted by \( \tilde{0} \).

Here \( F_\emptyset (e) = \tilde{0} \) for every \( e \in E \).
Definition 2.8 [13]
Let \( F_E \subseteq FS(U, E) \) and \( F_E(e) = \tilde{I} \) for all \( e \in E \).
Then \( F_E \) is called absolute fuzzy soft set. It is denoted by \( \tilde{E} \).

Definition 2.9 [13]
Let \( F_A, G_B \subseteq FS(U, E) \). If \( F_A(e) \subseteq G_B(e) \) for all \( e \in E \); i.e., if \( \mu \subseteq G_B \) for all \( e \in E \), i.e., if \( \mu \subseteq G_B \) for all \( x \in U \) and for all \( e \in E \), then \( F_A \) is said to be fuzzy soft subset of \( G_B \), denoted by \( F_A \subseteq G_B \).

Definition 2.10 [13]
Let \( F_A, G_B \subseteq FS(U, E) \). Then the union of \( F_A \) and \( G_B \) is also fuzzy soft set \( H_C \), defined by \( H_C(e) = \mu + H_C = \mu \cup G_B \) for all \( e \in E \) where \( C = A \cup B \). Here we write \( H_C = F_A \cup G_B \).

Definition 2.11 [13]
Let \( F_A, G_B \subseteq FS(U, E) \). Then the intersection of \( F_A \) and \( G_B \) is also a fuzzy soft set \( H_C \), defined by \( H_C(e) = \mu \cap H_C = \mu \cap G_B \) for all \( e \in E \) where \( C = A \cap B \). Here we write \( H_C = F_A \cap G_B \).

Definition 2.12
Let \( F_A \subseteq FS(U, E) \). The complement of \( F_A \) is denoted by \( F_A \) and is defined by \( F_A : E \rightarrow \tilde{P}(U) \) is a mapping given by \( F_A : (e) = [F(e)]^c \), \( \forall e \in E \).

### III. \( \alpha \)-LACUNARY \( \Delta \)-STATISTICALLY CONVERGENT

#### 3.1 Definition
Let us denote the fuzzy soft real number \( \tilde{r}_e \) where \( r \in R \) and \( \tilde{r} : E \rightarrow I^R \) where \( I^R \) is the set of all fuzzy sets on \( R \) and \( E \) is the parameter set. Denote fuzzy soft real number by \( \tilde{r} \). \( \tilde{r} \) is called the \( \alpha \)-level set of \( \tilde{r} \) corresponding to the parameter \( e \in E \) and is defined as \( \tilde{r} \) where \( E \subseteq R \).

#### 3.2 Definition
Let \( \tilde{t}_n \) be a sequence of fuzzy soft numbers and its corresponding \( \alpha \)-level sequence be \( \tilde{t}_n \) then \( \tilde{t}_n \) is said to be \( \Delta \)-bounded if \( \{ \Delta \tilde{t}_n \} \) is bounded subset of real numbers and is said to be \( \Delta \)-convergent to a real number \( \alpha \) if \( \lim_{n \to \infty} \Delta \tilde{t}_n = \alpha \), that is for every \( \varepsilon > 0 \) there exists a positive integer \( n_0 \) such that \( | \Delta \tilde{t}_n - \alpha | < \varepsilon \) for all \( n > n_0 \) provided \( \Delta \tilde{t}_n = [\tilde{t}_{n+1}]_\alpha - [\tilde{t}_n]_\alpha \). Let us denote \( \Delta \)-bounded and \( \Delta \)-convergent of \( \alpha \)-level fuzzy soft numbers to be \( \Delta^e \) and \( \Delta^a \).

#### 3.3 Definition
Let \( \theta = (k_i) \) be a lacunary sequence and let \( \tilde{t}_n = [\tilde{t}_n]_\alpha \) be a \( \alpha \)-level sequence of fuzzy soft real numbers then \( \tilde{t}_n \) is said to be \( \alpha \)-lacunary \( \Delta \)-statistically convergent to a real number \( t \) if for every \( \varepsilon > 0 \) \( \lim_{r \to \infty} \frac{1}{h_r} \{ k \in I_r : | \Delta \tilde{t}_n - t | \geq \varepsilon \} = 0 \).

Let us denote it by \( S_\theta \lim \Delta \tilde{t}_n = t \). Let us denote the set of all \( \alpha \)-lacunary \( \Delta \)-statistically convergent sequence be denoted \( S(\Delta^a) \).

#### 3.1 Theorem
Let \( \{ \tilde{s}_n \} \) be a sequence of fuzzy soft real numbers

1. If \( S_\theta \lim \Delta \tilde{t}_n = t \) and \( c \in R \) then \( S_\theta \lim c \Delta \tilde{t}_n = ct \).
2. If \( S_\theta \lim \Delta \tilde{t}_n = s \) and \( S_\theta \lim \Delta \tilde{t}_n = t \) then \( S_\theta \lim (\Delta \tilde{t}_n + \Delta \tilde{t}_n) = s + t \).

**Proof:**
For \( \alpha \in [0, 1] \) and \( c \in R \) then
\[
|c \Delta \tilde{t}_n - c t| = |c (\Delta \tilde{t}_n - t)| \geq |c t| \quad \Delta \tilde{t}_n - t |t|
\]
For given \( \varepsilon > 0 \) we have
\[
\frac{1}{h_r} \{ k \in I_r : |c \Delta \tilde{t}_n - c t | \geq \varepsilon \}
\]
\[
\frac{1}{h_r} \left\{ k \in I_r : \left| \Delta [\tilde{s}_n]_{e,a} - t \right| \geq \varepsilon \right\} = 1
\]

From (1) we get \( S_\theta - \lim c \Delta [\tilde{s}_n]_{e,a} = ct \)

(2) suppose that
\( S_\theta - \lim \Delta [\tilde{s}_n]_{e,a} = s \) and
\( S_\theta - \lim \Delta [\tilde{t}_n]_{e,a} = t \)

By Minkowski’s inequality we get,
\[
\left| \Delta [\tilde{s}_n]_{e,a} + \Delta [\tilde{t}_n]_{e,a} \right| - (s+t) \leq \left| \Delta [\tilde{s}_n]_{e,a} - s \right| + \left| \Delta [\tilde{t}_n]_{e,a} - t \right|
\]

Therefore given \( \varepsilon > 0 \) we have,
\[
\frac{1}{h_r} \left\{ k \in I_r : \left| \Delta [\tilde{s}_n]_{e,a} + \Delta [\tilde{t}_n]_{e,a} - (s+t) \right| \geq \varepsilon \right\}
\]

Now consider the \( k_m \)-th term of the statistical limit expression
\[
\frac{1}{n} \left\{ k \leq n : \left| [\tilde{t}_n]_{e,a} - t \right| \geq \varepsilon \right\} \Rightarrow \frac{1}{k_m} \left\{ k \in \bigcup_{k=1}^m I_r : \left| [\tilde{t}_n]_{e,a} - t \right| \geq \varepsilon \right\}
\]

Hence
\( S_\theta - \lim (\Delta [\tilde{t}_n]_{e,a}) = s \)

3.2 Theorem
If \( \{s_n\} \subseteq s(\Delta^\alpha) \cup s_\theta (\Delta^\alpha) \) then we have
\( S_\theta - \lim (\Delta [\tilde{t}_n]_{e,a}) = s - \lim (\Delta [\tilde{s}_n]_{e,a}) \)

Proof
Suppose that \( s - \lim (\Delta [\tilde{s}_n]_{e,a}) = t \) and
\( S_\theta - \lim (\Delta [\tilde{t}_n]_{e,a}) = t \) and \( t \neq \bar{t} \)

Then we have \( \left| t - \bar{t} \right| > 0 \)
Suppose \( \left| t - \bar{t} \right| / 2 > \varepsilon > 0 \)

We have
\[
\lim_{n \to \infty} \frac{1}{n} \left\{ k \leq n : \left| [\tilde{t}_n]_{e,a} - t \right| \geq \varepsilon \right\} = 1
\]
\[ |\Delta [\tilde{t}_n]_{e,a} - \Delta [\tilde{t}_m]_{e,a}| < \varepsilon \text{ for } n, m \geq n_0 \]

As \(d(x, y) = |x - y|\) is a standard metric on real numbers \(R\) and \(R\) is complete with this metric, \(\{\Delta [\tilde{t}_n]_{e,a}\}\) is a Cauchy sequence in \(R\). And so it converges to \(t \in R\)

\[
\text{(ie) } \lim_{n \to \infty} \Delta [\tilde{t}_n]_{e,a} = t
\]

This implies \(|\Delta [\tilde{t}_n]_{e,a} - t| < \varepsilon\) for all \(n \geq n_0\).

Again consider (1)

\[
D([\tilde{t}_n]_{e,a}, [\tilde{t}_m]_{e,a}) = \inf \{d(\Delta [\tilde{t}_n]_{e,a}, \Delta [\tilde{t}_m]_{e,a}) \}
\]

Allow \(m \to \infty\) then

\[
\lim_{m \to \infty} D([\tilde{t}_n]_{e,a}, [\tilde{t}_m]_{e,a}) = \inf \{d(\Delta [\tilde{t}_n]_{e,a}, t)\} \to 0
\]

as \(n \to \infty\)

Hence

\[
D([\tilde{t}_n]_{e,a}, [\tilde{t}_m]_{e,a}) \to 0 \text{ as } n \to \infty
\]

Thus \(\Delta^\alpha\) is complete and by similar argument \(\Delta^\varepsilon\) is complete.

### 3.4 Theorem

Let \(\{\tilde{t}_n\}\) be a sequence of fuzzy soft real numbers such that \(S_\theta - \lim(\Delta [\tilde{t}_n]_{e,a}) = t\) exists then it is unique.

**Proof**

Suppose there exists \(t_1, t_2\) with \(t_1 \neq t_2\) such that

\[
S_\theta - \lim(\Delta [\tilde{t}_n]_{e,a}) = t_1 \text{ and } S_\theta - \lim(\Delta [\tilde{t}_n]_{e,a}) = t_2.
\]

Then for every \(\varepsilon > 0\),

\[
\lim_{r \to \infty} \frac{1}{h_r} \{k \in I_r : |\Delta [\tilde{t}_n]_{e,a} - t_1| \geq \varepsilon \} = 0 \text{ and }
\]

\[
\lim_{r \to \infty} \frac{1}{h_r} \{k \in I_r : |\Delta [\tilde{t}_n]_{e,a} - t_2| \geq \varepsilon \} = 0
\]

This implies

\[
|\Delta [\tilde{t}_n]_{e,a} - t_1| < \varepsilon \text{ and } |\Delta [\tilde{t}_n]_{e,a} - t_2| < \varepsilon
\]

Also

\[
|t_1 - t_2| = |\Delta [\tilde{t}_n]_{e,a} - t_1 + t_2 - \Delta [\tilde{t}_n]_{e,a}| \\
\leq | -t_1 + \Delta [\tilde{t}_n]_{e,a}| + |\Delta [\tilde{t}_n]_{e,a} - t_2| < 2 \varepsilon
\]

Where \(\varepsilon\) is arbitrary

Hence \(t_1 = t_2\).

### IV. CONCLUSION

In this paper we have defined new concepts \(\Delta\)-convergent, \(\Delta\)-bounded and \(\alpha\)-lacunary \(\Delta\)-statistically convergent sequence in the space of fuzzy soft real numbers and related to this some basic theorem are proved.

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