On the Cubic Equation with Four Unknowns

\[ 2x^3 + 8z^3 = 2y^3 + 8w^3 + 12(x - y)^3 \]

R. Anbuselvi  
Associate Professor, Department of Mathematics,  
A.D.M. College for women, Nagapattinam, India  
anbuselvi134@gmail.com

K. S. Araththi  
Research Scholar, Department of Mathematics,  
A.D.M college for Women, Nagapattinam, India  
ksaraththi@gmail.com

Abstract

The cubic equation \[ 2x^3 + 8z^3 = 2y^3 + 8w^3 + 12(x - y)^3 \] is analyzed for its non-zero distinct integer solutions. Five different patterns of non-zero distinct integer solutions to the equation under consideration are obtained. A few interesting relation between the solutions and special numbers are exhibited.

Keywords: Integral solutions, Ternary Cubic.

I. INTRODUCTION

The Cubic Equation offers an unlimited field for research because of their variety [1-2]. For an extensive review of various problems, one may refer [3-10]. This communication concerns with yet another interesting Ternary Quadratic equation

\[ 2x^3 + 8z^3 = 2y^3 + 8w^3 + 12(x - y)^3 \] representing a homogenous cone for determining its infinitely many non-zero integral solutions. Also a few interesting relations among the solutions have been presented.

II. NOTATIONS

obl: Oblong number of rank ‘n’  
t:\[ t_{mn} = \text{Polygonal number of rank ‘n’ with sides’m} \]

III. METHOD OF ANALYSIS

The cubic Diophantine equation with four unknowns to be solved for getting non-zero integral solution is

\[ 2x^3 + 8z^3 = 2y^3 + 8w^3 + 12(x - y)^3 \] ---- (1)

On substituting the linear transformations

\[ v = \frac{2}{3} pq \]

\[ s = \frac{p^2 - q^2}{2} \]

We obtain five different patterns of integral solutions to (1) through solving (3) which are illustrated as follows:

Pattern 1:

In (1), we get

\[ 12u^2v = 48s^2v + 6v^3 + 96v^3 + 6v^3 \]

\[ u^2 = 4s^2 + 9v^2 \] ---- (3)

Which is in the form of famous Pythagorean equation.

\[ x = u + v, \quad y = u - v, \quad W = s + v, \quad z = s - v \] ---- (2)
\[ s = \frac{p^2 - q^2}{2} \]
\[ = 18(p^2 - q^2) \]

\( x = u + v \)
\[ = 36p^2 + 36q^2 + 24pq \]
\[ = 36(p^2 + q^2) + 24pq \]

\( y = u - v \)
\[ = 36p^2 + 36q^2 - 24pq \]
\[ = 36(p^2 + q^2) - 24pq \]

\( z = s - v \)
\[ = 18p^2 - 18q^2 - 24pq \]
\[ = 18(p^2 - q^2) - 24pq \]

\( w = s + v \)
\[ = 18p^2 - 18q^2 + 24pq \]
\[ = 18(p^2 - q^2) + 24pq \]

Now, we get
\[ x = x(p, q) = 36(p^2 + q^2) + 24pq \]
\[ y = y(p, q) = 36(p^2 + q^2) - 24pq \]
\[ z = z(p, q) = 18(p^2 - q^2) - 24pq \]
\[ w = w(p, q) = 18(p^2 - q^2) + 24pq \]

Properties

(i) \([x(p, q) + y(p, q)]\) is a nasty number
(ii) \([x(p, q) + z(p, q)]\) is a nasty number
(iii) \([x(p, q) + w(p, q)] - 6\alpha_{20, \alpha} \) is a nasty number
(iv) \([y(p, q) + w(p, q)]\) is a nasty number

**pattern 2:**

Let
\[ 2s = 2pq \implies s = pq \]
\[ v = \frac{p^2 - q^2}{3} \text{ and } u = p^2 + q^2 \]

Put \( p = 3p \) and \( q = 3q \), we get
\[ u = 3(p^2 + q^2) \]
\[ v = (p^2 - q^2) \]
\[ s = 9pq \]

In view of (2) the non-zero integer solutions to (1) are given by
\[ x = x(p, q) = 4p^2 + 2q^2 \]
\[ y = y(p, q) = 2p^2 - 2q^2 \]
\[ z = z(p, q) = 9pq - p^2 + q^2 \]
\[ w = w(p, q) = p^2 - q^2 + 9pq \]

Properties

(i) \([x(p, q) + y(p, q)]\) is a nasty number
(ii) \([x(p, q) + z(p, q)]\) is a nasty number
(iii) \([x(p, q) + w(p, q)]\) is a nasty number
(iv) \([y(p, q) + z(p, q)]\) is a nasty number
(v) \([y(p, q) + w(p, q)]\) is a nasty number

**pattern 3:**

Equation (3) can be re-written as
\[ u^2 - (3v)^2 = (2s)^2 \]

which is written in the form of ratio as,
\[
\frac{u + 3v}{2s} = \frac{2s}{u - 3v} = \frac{\alpha}{\beta} \quad \cdots (4)
\]

Which is equivalent to the system of equations,

\[
\begin{align*}
u\beta + 3v\beta - 2s\alpha &= 0 \\
-u\alpha + 3av + 2\beta s &= 0
\end{align*}
\]

Applying the method of cross multiplication we have,

\[
\begin{align*}
u &= 6\alpha^2 + 6\beta^2 \\
v &= 2\alpha^2 - 2\beta^2 \\
s &= 6\alpha\beta
\end{align*}
\]

In view of (2) the non-zero integer solutions to (1) are given by

\[
\begin{align*}
x &= x(\alpha, \beta) = 8\alpha^2 + 4\beta^2 \\
y &= y(\alpha, \beta) = 4\alpha^2 + 8\beta^2 \\
w(\alpha, \beta) &= 2\alpha^2 - 2\beta^2 + 6\alpha\beta \\
z &= z(\alpha, \beta) = 2\beta^2 - 2\alpha^2 + 6\alpha\beta
\end{align*}
\]

Properties

(i) \[x(\alpha,1) + y(\alpha,1) - 12 = \text{a nasty number}\]
(ii) \[[x(\alpha,1) + y(\alpha,1)] \equiv 0 \pmod{2}\]
(iii) \[[x(\alpha,1) - w(\alpha,1)] \equiv 0 \pmod{3}\]
(iv) \[[w(\alpha,1) + z(\alpha,1)] \equiv 0 \pmod{4}\]
(v) \[[y(\alpha,1) + z(\alpha,1)] \equiv 0 \pmod{2}\]

Pattern 4:

Equation (3) can also be equivalent to the system of equations,

\[
\begin{align*}
u\beta + 2s\beta - \alpha v &= 0 \\
-u\alpha + 2s\alpha + 9v\beta &= 0
\end{align*}
\]

Applying the method of cross multiplication we have,

\[
\begin{align*}
u &= 2\alpha^2 + 18\beta^2 \\
s &= \alpha^2 - 9\beta^2 \\
v &= 4\alpha\beta
\end{align*}
\]

In view of (2) the non-zero integer solutions to (1) are given by

\[
\begin{align*}
x &= x(\alpha, \beta) = 2\alpha^2 + 18\beta^2 + 4\alpha\beta \\
y &= y(\alpha, \beta) = 2\alpha^2 + 18\beta^2 - 4\alpha\beta
\end{align*}
\]
\[ w = w(\alpha, \beta) = \alpha^2 - 9\beta^2 + 4\alpha\beta \]

**Properties**

(i) \[ [x(\alpha, 1) - y(\alpha, 1)] \equiv 0 \pmod{8} \]
(ii) \[ [x(\alpha, 1) + w(\alpha, 1)] \equiv 0 \pmod{3} \]
(iii) \[ 6x(\alpha, 1) - 3w(\alpha, 1) \equiv 0 \pmod{3} \]
(iv) \[ [w(\alpha, 1) - z(\alpha, 1)] \equiv 0 \pmod{8} \]
(v) \[ [y(\alpha, 1) + w(\alpha, 1)] \equiv 0 \pmod{3} \]

**IV. CONCLUSION**

To conclude, one may search for other patterns of solutions to the equation under consideration.

**REFERENCES**


